ON ESTIMATES OF THE SOLUTIONS OF SYSTEMS OF DIFFERENTIAL EQUATIONS OF THE ACCUMULATION OF DISTURBANCES AND THE STABILITY OF MOTION OVER A FINITE TIME INTERVAL

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Problems on the stability of motion over a finite time interval are based on the estimates of the solutions of systems of differential equations. In this paper there is given a method for estimating the solutions in certain cases. The conditions for stability are also determined.

1. Linear system. Let us consider the system

$$\frac{dx_s}{dt} = p_{s1}(t)x_1 + \cdots + p_{sn}(t)x_n \qquad (s = 1, \ldots, n) \tag{1.1}$$

Here $p_{sr}(t)$ is a real, bounded, continuous function of time t. The characteristic equation of (1.1) has the form:

$$|p_{sr}(t) - \delta_{sr}\varkappa| = 0$$

Let us suppose that this equation has simple roots only. Of these roots, let there be m real ones $\varkappa_j(t)$ $(j = 1, \ldots, m)$ and 2σ complex ones $\lambda_i(t) \pm \mu_i(t)\sqrt{-1}$ $(i = m + 1, \ldots, n - \sigma)$. As is well known, there exists a linear nonsingular transformation with variable coefficients

$$y_s = a_{s1}(t)x_1 + \cdots + a_{sn}(t)x_n \qquad (s = 1, \ldots, n) \qquad |a_{sr}(t)| \neq 0 \tag{1.2}$$

and an inverse transformation

$$x_s = b_{s1}(t)y_1 + \cdots + b_{sn}(t)y_n \qquad (s = 1, \ldots, n) \qquad |b_{sr}(t)| \neq 0 \tag{1.3}$$

with the aid of which the system (1.1) can be reduced to the canonical form

$$\frac{dy_j}{dt} = \varkappa_j y_j + Q_j \qquad (j = 1, \ldots, m)$$

$$\frac{dy_i}{dt} = \lambda_i y_i - \mu_i y_{\sigma+i} + Q_i \qquad (i = m+1, \ldots, n-\sigma) \tag{1.4}$$

$$\frac{dy_{\sigma+i}}{dt} = \lambda_i y_{\sigma+i} + \mu_i y_i + Q_{\sigma+i}$$

Here the $Q_s (s = 1, \ldots, n)$ are linear combinations of the variables y_s with known coefficients, depending on the coefficients of the transformations (1.2) and (1.3).

Let us consider the function

$$V = e^{-\alpha(t)} \, (y_1^2 + \cdots + y_n^2) \tag{1.5}$$

where $\alpha(t)$ and its derivative are real bounded (in some region) continuous functions. The function $\alpha(t)$ is still undetermined.

The derivative dV/dt can be found on the basis of (1.4) in the form

$$\frac{dV}{dt} = e^{-\alpha(t)} \left[\sum_{j=1}^{m} (2\varkappa_j - \alpha') y_j^2 + \sum_{i=m+1}^{n-\sigma} (\lambda_i - \alpha')(y_i^2 + y_{\sigma+i}^2) + \sum_{s=1}^{n} 2 y_s Q_s\right] \tag{1.6}$$

We denote the function occurring within the brackets by

$$H = \sum_{i,j} h_{ij} y_i y_j = \sum_{i,j} (h_{ij}^* - \delta_{ij}\alpha') \, y_i y_j \qquad \begin{pmatrix} h_{ij} = h_{ji} \\ h_{ij}^* = h_{ji}^* \end{pmatrix} \tag{1.7}$$

In order that $dV/dt \leqslant 0$, the quadratic form H has to be non-positive. The characteristic equation of this form is

$$| h_{ij}^* - \delta_{ij} (\alpha' + \rho) | = 0 \tag{1.8}$$

Therefore, we may write

$$H = \rho_1 \xi_1^2 + \cdots + \rho_n \xi_n^2 \tag{1.9}$$

where $\rho_1, \ldots, \rho_n$ are the roots of the equation (1.8). We introduce the notation $\alpha' + \rho = -\psi$. Then equation (1.8) takes on the form

$$| h_{ij}^* + \delta_{ij} \psi | = 0 \tag{1.10}$$

Let $\psi_1, \ldots, \psi_n$ be the roots of this equation. Then equation (1.9) can be rewritten in the form

$$H = (-\psi_1 - \alpha') \xi_1^2 + \cdots + (-\psi_n - \alpha') \xi_n^2$$

The form H will be non-positive if

$$-\psi_k - \alpha' \leqslant 0 \tag{1.11}$$

where ψ_k is the smallest one of the roots $\psi_1, \ldots, \psi_n$ of equation (1.10). From (1.11) we obtain

$$-\alpha(t) \leqslant \int \psi_k(t)\,dt + C \tag{1.12}$$

where C is an arbitrary constant.

Under condition (1.12) the derivative $dV/dt \leqslant 0$, and, hence,

$$V \leqslant V_0 \tag{1.13}$$

Here V_0 is the value of V when $t = t_0$. Let us substitute (1.12) into (1.5). Obviously, in view of (1.13), the value of C will be immaterial, and one may let $C = 0$. Thus, according to (1.5) we have

$$V = (y_1^2 + \cdots + y_n^2) \exp \int \psi_k(t)\,dt \tag{1.14}$$

Let us suppose that when $t = t_0$, we have

$$|x_{s0}| \leqslant x_{s0}^{\circ} \qquad (s = 1, \ldots, n) \tag{1.15}$$

Then, when $t = t_0$, we have in accordance with (1.2)

$$|y_{s0}| = |a_{s1}(t_0)|\, x_{10}^{\circ} + \cdots + |a_{sn}(t_0)|\, x_{n0}^{\circ} \qquad (s = 1, \ldots, n) \tag{1.16}$$

If $t > t_0$, we have, because of (1.13), the following inequality

$$(y_1^2 + \cdots + y_n^2) \exp \int \psi_k(t)\,dt \leqslant (y_{10}^2 + \cdots + y_{n0}^2) \exp \int \psi_k(t)\,dt \,\Big|_{t_0}$$

or

$$y_1^2 + \cdots + y_n^2 \leqslant A \exp - \int_{t_0}^{t} \psi_k(\xi)\,d\xi \qquad (A = y_{10}^2 + \cdots + y_{n0}^2) \tag{1.17}$$

From this it follows that

$$|y_s| \leqslant A^{1/2} \exp\left(-\frac{1}{2}\int_{t_0}^{t} \psi_k(\xi)\,d\xi\right) \qquad (s = 1, \ldots, n) \tag{1.18}$$

Finally, in accordance with (1.3), we obtain an estimate of the solution of the system (1.1):

$$|x_s| \leqslant A^{1/2}\,[|b_{s1}(t)| + \cdots + |b_{sn}(t)|] \exp\left(-\frac{1}{2}\int_{t_0}^{t} \psi_k(\xi)\,d\xi\right) \qquad (s = 1, \ldots, n) \tag{1.19}$$

In order to satisfy, under condition (1.15), the inequality

$$|x_s| \leqslant x_s^{\circ} \qquad (s = 1, \ldots, n) \tag{1.20}$$

for all t in the interval $t_0 \leqslant t \leqslant T$, where x_s^0, T are given numbers, it is sufficient that

$$A^{1/2}[|b_{s1}(t)|+\cdots+b_{sn}(t)|]\exp\Big(-\frac{1}{2}\int_{t_0}^{t}\psi_k(\xi)\,d\xi\Big)\leqslant x_s^0 \quad \binom{t_0\leqslant t\leqslant T}{s=1,\ldots,n} \qquad (1.21)$$

This is the condition of stability of the motion during the finite time interval $[t_0, T]$.

2. System with slowly changing coefficients. In this case the coefficients of the system (1.1) have the form

$$p_{sr}(t)=C_{sr}+\varepsilon f_{sr}(t) \qquad (s, r=1,\ldots,n) \qquad (2.1)$$

where C_{sr} are constants, ϵ is a sufficiently small number and f_{sr} are bounded functions. The system (1.1) in this case takes on the form

$$\frac{dx_s}{dt}=C_{s1}x_1+\cdots+C_{sn}x_n+\varepsilon(f_{s1}x_1+\cdots+f_{sn}x_n) \qquad (s=1,\ldots,n) \qquad (2.2)$$

We introduce the nonsingular linear transformation with constant coefficients

$$y_s=a_{s1}x_1+\cdots+a_{sn}x_n \qquad (s=1,\ldots,n) \qquad (2.3)$$

The inverse transformation

$$x_s=b_{s1}y_1+\cdots+b_{sn}y_n \qquad (s=1,\ldots,n) \qquad (2.4)$$

will be nonsingular also.

Let us consider the following cases.

1. The roots of the equation

$$|C_{sr}-\delta_{sr}\lambda|=0 \qquad (2.5)$$

are simple and real.

In this case the system (2.2) can be reduced with the aid of the transformation (2.3) to the form

$$\frac{dy_s}{dt}=\lambda_s y_s+\varepsilon Q_s \qquad \Big(Q_s=\sum_k f_{sr}(b_{k1}y_1+\cdots+b_{kn}y_n)\Big) \qquad (s=1,\ldots,n) \qquad (2.6)$$

Let us consider the function

$$V=e^{-\alpha_1(t)}y_1^2+\cdots+e^{-\alpha_n(t)}y_n^2 \qquad (2.7)$$

Here $\alpha_1(t), \ldots, \alpha_n(t)$ and their derivatives are real, bounded (in some region) and continuous functions. We compute dV/dt. In view of (2.6) we obtain

$$\frac{dV}{dt}=\{e^{-\alpha_1(t)}(-\alpha_1'+2\lambda_1)y_1^2+\cdots+e^{-\alpha_n(t)}(-\alpha_n'+2\lambda_n)y_n^2\}+$$
$$+\varepsilon[e^{-\alpha_1(t)}2y_1Q_1+\cdots+e^{-\alpha_n(t)}2y_nQ_n] \qquad (2.8)$$

Since the quantity ϵ is assumed to be sufficiently small, and the function in the square bracket is bounded, the sign of dV/dt will be definite. If

$$-\alpha_s' + 2\lambda_s < 0 \qquad (s = 1, \ldots, n) \tag{2.9}$$

then $dV/dt < 0$. The condition (2.9) can be expressed in the form

$$-\alpha_s'(t) \leqslant -2\lambda_s - \delta \qquad (\delta > 0) \qquad (s = 1, \ldots, n) \tag{2.10}$$

Integrating this, one obtains

$$-\alpha_s(t) \leqslant -2\lambda_s t - \delta t + C_s \qquad (s = 1, \ldots, n) \tag{2.11}$$

Here the arbitrary constants C_s can be assumed to be zero (analogous to Section 1).

In accordance with (2.11) it is now seen that (2.7) can be written in the form

$$V = e^{-\delta t}(e^{-2\lambda_1 t} y_1^2 + \cdots + e^{-2\lambda_n t} y_n^2) \tag{2.12}$$

Herein we shall have $V < V_0$, or (assuming for the sake of simplicity that $t_0 = 0$)

$$e^{-\delta t}(e^{-2\lambda_1 t} y_1^2 + \cdots + e^{-2\lambda_n t} y_n^2) < (y_{10}^2 + \cdots + y_{n0}^2) = A$$

Whence

$$|y_s| < A^{1/2} e^{(\lambda_s + \delta/2)t} = e^{1/2\delta t} A^{1/2} e^{\lambda_s t} \qquad (s = 1, \ldots, n) \tag{2.13}$$

Finally, we obtain an estimate for the solutions of the system (2.2):

$$|x_s| < e^{1/2\delta t} A^{1/2} [|b_{s1}| e^{\lambda_1 t} + \cdots + |b_{sn}| e^{\lambda_n t}] \qquad (s = 1, \ldots, n) \tag{2.14}$$

The condition of stability will be (2.15)

$$e^{1/2\delta t} A^{1/2} [|b_{s1}| e^{\lambda_1 t} + \cdots + |b_{sn}| e^{\lambda_n t}] \leqslant x_s^\circ \qquad (t_0 \leqslant t \leqslant T) \qquad (s = 1, \ldots, n)$$

2. The equation (2.5) has m simple real roots λ_j ($j = 1, \ldots, m$) and 2σ simple complex roots $\lambda_i \pm \mu_i \sqrt{-1}$ ($k = m + 1, \ldots, n - \sigma$).

In this case the canonical form of the system (2.2) will be

$$\frac{dy_j}{dt} = \lambda_j y_j + \varepsilon Q_j \qquad (j = 1, \ldots, m) \tag{2.16}$$

$$\begin{aligned} \frac{dy_i}{dt} &= \lambda_i y_i - \mu_i y_{\sigma+i} + \varepsilon Q_i \\ \frac{dy_{\sigma+i}}{dt} &= \lambda_i y_{\sigma+i} + \mu_i y_i + \varepsilon Q_{\sigma+i} \end{aligned} \qquad (i = m + 1, \ldots, n - \sigma)$$

Let us consider the function

$$V = e^{-\alpha_1(t)} y_1^2 + \cdots + e^{-\alpha_m(t)} y_m^2 + \sum_{i=m+1}^{n-\sigma} e^{-\alpha_i(t)} (y_i^2 + y_{\sigma+i}^2) \qquad (2.17)$$

On the basis of (2.16), the derivative dV/dt can be written as

$$\frac{dV}{dt} = \sum_{j=1}^{m} e^{-\alpha_j(t)} (-\alpha_j' + 2\lambda_j) y_j^2 +$$

$$+ \sum_{i=m+1}^{n-\sigma} e^{-\alpha_i(t)} (-\alpha_i' + 2\lambda_i)(y_i^2 + y_{\sigma+i}^2) + \varepsilon \sum_{s=1}^{n} 2y_s Q_s \qquad (2.18)$$

If the conditions

$$\begin{aligned} &-\alpha_j'(t) + 2\lambda_j < 0 \qquad (j = 1, \ldots, m) \\ &-\alpha_i'(t) + 2\lambda_i < 0 \qquad (i = m+1, \ldots, n-\sigma) \end{aligned} \qquad (2.19)$$

are satisfied, then $dV/dt < 0$ and, hence, $V < V_0$.

Analogous to case 1, the function V can be represented in the form

$$V = e^{-\delta t} \Big[\sum_{j=1}^{m} e^{-2\lambda_j t} y_j^2 + \sum_{i=m+1}^{n-\sigma} e^{-2\lambda_i t} (y_i^2 + y_{\sigma+i}^2)\Big] \qquad (2.20)$$

Since $V < V_0$, we have (under the assumption that $t_0 = 0$)

$$e^{-\delta t} \Big[\sum_{j=1}^{m} e^{-2\lambda_j t} y_j^2 + \sum_{i=m+1}^{n-\sigma} e^{-2\lambda_i t} (y_i^2 + y_{\sigma+i}^2)\Big] < \sum_{s=1}^{n} y_{s0}^2 = A \qquad (2.21)$$

Finally, on the basis of (2.21), we obtain

$$\begin{aligned} &| y_j | < e^{1/2\,\delta t} A^{1/2} e^{\lambda_j t} \qquad (j = 1, \ldots, m) \qquad (2.22) \\ &| y_i | < e^{1/2\,\delta t} A^{1/2} e^{\lambda_i t}, \qquad | y_{\sigma+i} | < e^{1/2\,\delta t} A^{1/2} e^{\lambda_i (t)} \qquad (i = m+1, \ldots, n-\sigma) \end{aligned}$$

and, hence, in accordance with (2.4),

$$| x_s | < e^{1/2\,\delta t} A^{1/2} [\, | b_{s1} | e^{\lambda_1 t} + \cdots + | b_{sm} | e^{\lambda_m t} +$$

$$+ | b_{sm+1} | e^{\lambda_{m+1} t} + \cdots + | b_{sn} | e^{\lambda_{n-\sigma} t}] \qquad (s = 1, \ldots, n) \qquad (2.23)$$

Thus, the condition of stability will be

$$e^{1/2\,\delta t} A^{1/2} [\, | b_{s1} | e^{\lambda_1 t} + \cdots + | b_{sm} | e^{\lambda_m t} +$$

$$+ | b_{sm+1} | e^{\lambda_{m+1} t} + \cdots + | b_{sn} | e^{\lambda_{n-\sigma} t}] \leqslant x_s^\circ \quad (t_0 \leqslant t \leqslant T) \quad (s = 1, \ldots, n) \qquad (2.24)$$

The investigation of the case of multiple roots can be carried out in a similar way.

3. System with continuously acting disturbances. In this case the equations of motion for the system are

$$\frac{dx_s}{dt} = p_{s1}(t)x_1 + \cdots + p_{sn}(t)x_n + R_s \qquad (s = 1, \ldots, n) \tag{3.1}$$

Here the functions R_s describe continuously acting disturbances. In what follows we shall consider two cases:

$$|R_s| \leqslant R_s^\circ(t) \qquad (s = 1, \ldots, n) \tag{3.2}$$

$$|R_s| \leqslant l_s \qquad (s = 1, \ldots, n) \tag{3.3}$$

where the $R_s^0(t)$ are known functions and the l_s are constants.

Let us estimate the solutions of the system (3.1). As is well known, the general solution of the system (3.1) has the form:

$$x_s^* = x_s + u_s \qquad (s = 1, \ldots, n) \tag{3.4}$$

Here x_s is the complementary function of the homogeneous equation (1.1), while u_s is the particular integral of system (3.1). An estimate of x_s is given in Section 1. Therefore, we shall look for an estimate of u_s.

Let $x_1^{(l)}, \ldots, x_n^{(l)}$ $(l = 1, \ldots, n)$ be a fundamental system of solutions of the homogeneous equation, where for $t = t_0$

$$x_s^{(l)}(t_0) = \begin{cases} 1 & (l = s) \\ 0 & (l \neq s) \end{cases} \tag{3.5}$$

By Lagrange's method, a particular solution of the system (3.1) will be

$$u_s = \sum_{i=1}^{n} x_s^{(i)}(t) \int\limits_{\tau=t_0}^{t} \frac{1}{D(\tau)} \sum_{l=1}^{n} D_{li}(\tau) R_l(\tau)\, d\tau \qquad (s = 1, \ldots, n) \tag{3.6}$$

Here $D = \det \| x_s^{(l)} \|$, while D_{li} is the minor of $x_l^{(i)}$ with the proper sign. One may rewrite (3.6) in the form

$$u_s = \sum_{l=1}^{n} \int\limits_{\tau=t_0}^{t} \frac{1}{D(\tau)} \sum_{i=1}^{n} x_s^{(i)}(t) D_{li}(\tau) R_l(\tau)\, d\tau \tag{3.7}$$

Let us introduce the symbol $Z_s^{(l)}(t, \tau)$:

$$Z_s^{(l)}(t, \tau) = \frac{1}{D(\tau)} \sum_{i=1}^{n} x_s^{(i)}(t) D_{li}(\tau) \qquad (s, l = 1, \ldots, n) \tag{3.8}$$

It is not difficult to prove that the functions $Z_s^{(l)}(t, \tau)$ constitute a fundamental system of solutions of the homogeneous equation.

Indeed, from (3.8) it is seen that the $Z_s^{(l)}$ are linear combinations of the solutions $x_s^{(l)}$ with the coefficients $D_{li}(\tau)/D(\tau)$. Furthermore, from (3.8) it follows that when $t = \tau$ and $t = t_0$, we have

$$Z_s^{(l)}(\tau, \tau) = \begin{cases} 1 & (l = s) \\ 0 & (l \neq s) \end{cases} \qquad Z_s^{(l)}(t_0, t_0) = \begin{cases} 1 & (l = s) \\ 0 & (l \neq s) \end{cases} \tag{3.9}$$

Therefore, we have the following estimates for $Z_s^{(l)}$ in accordance with (1.19):

$$|Z_s^{(l)}| \leqslant A^{1/2}[|b_{s1}(t)| + \cdots + |b_{sn}(t)|] \exp\left(-\frac{1}{2}\int_{t_0}^{t} \psi_k(\xi)\, d\xi\right) \quad (s = 1, \ldots, n) \tag{3.10}$$

which are obtained with the aid of a nonsingular linear transformation

$$y_s^{(l)} = a_{s1}(t) Z_1^{(l)}(t, \tau) + \cdots + a_{sn}(t) Z_n^{(l)}(t, \tau) \qquad (l, s = 1, \ldots, n) \tag{3.11}$$

and the inverse transformation

$$Z_s^{(l)}(t, \tau) = b_{s1}(t) y_1^{(l)} + \cdots + b_{sn}(t) y_n^{(l)} \qquad (l, s = 1, \ldots, n) \tag{3.12}$$

From this it can be seen that $A = y_{10}^{(l)2} + \ldots + y_{n0}^{(l)2}$. But according to (3.9) and (3.11) the $y_s^{(l)}$ are given as

$$y_{10}^{(l)} = a_{1l}(t_0), \ldots, y_{n0}^{(l)} = a_{nl}(t_0) \qquad (l = 1, \ldots, n) \tag{3.13}$$

Therefore, we have

$$y_{10}^{(l)2} + \cdots + y_{n0}^{(l)2} = a_{1l}^2(t_0) + \cdots + a_{nl}^2(t_0) \leqslant A_{\max} \qquad (l = 1, \ldots, n) \tag{3.14}$$

where

$$A_{\max} = \max\left\{\sum_{s=1}^{n} a_{s1}^2(t_0), \ldots, \sum_{s=1}^{n} a_{sn}^2(t_0)\right\}$$

Hence, from (3.10) we obtain

$$|Z_s^{(l)}| \leqslant A_{\max}^{1/2}[|b_{s1}(t)| + \cdots + |b_{sn}(t)|] \exp -\frac{1}{2}\int_{t_0}^{t} \psi_k(\xi)\, d\xi = Z_s^*(t)$$
$$(s = 1, \ldots, n) \tag{3.15}$$

We note that in the particular case

$$|y_{s0}^{(l)}| \leqslant |y_{s0}|, \qquad \text{or} \qquad |Z_{s0}^{(l)}| \leqslant |x_{s0}| \tag{3.16}$$

the inequalities (1.19) are valid for the estimates of $|Z_s^{(l)}|$.

Let us return to (3.7). Because of (3.8) we have

$$u_s = \sum_{l=1}^{n} \int_{\tau=t_0}^{t} Z_s^{(l)}(t,\tau) R_l(\tau)\, d\tau \qquad (s=1,\ldots,n) \tag{3.17}$$

By Holder's inequality we have

$$|u_s| \leqslant \sum_{l=1}^{n} \Big\{ \int_{\tau=t_0}^{t} |Z_s^{(l)}(t,\tau)|^p\, d\tau \Big\}^{1/p} \Big\{ \int_{\tau=t_0}^{t} |R_l(\tau)|^q\, d\tau \Big\}^{1/q} \qquad (s=1,\ldots,n) \tag{3.18}$$

Here

$$p > 1, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

By (3.2) we have

$$\int_{\tau=t_0}^{t} |R_l(\tau)|^q\, d\tau \leqslant \int_{\tau=t_0}^{t} (R_l^\circ(\tau))^q\, d\tau$$

Therefore, in accordance with (3.15), we obtain

$$|u_s| \leqslant Z_s^*(t)(t-t_0)^{\frac{1}{p}} \sum_{l=1}^{n} \Big\{ \int_{\tau=t_0}^{t} (R_l^\circ(\tau))^q\, d\tau \Big\}^{\frac{1}{q}} \qquad (s=1,\ldots,n) \tag{3.19}$$

For the case of (2.3) we have

$$|u_s| \leqslant Z_s^*(t)(t-t_0) \sum_{l=1}^{n} l_l \qquad (s=1,\ldots,n) \tag{3.20}$$

The final estimates of the complementary functions of system (3.1) can be found from (1.19), (3.15) and (3.19) in the form

$$|x_s^*| \leqslant [\,|b_{s1}(t)| + \cdots + |b_{sn}(t)|\,] \Big\{ A^{1/2} + A_{\max}^{1/2}(t-t_0)^{\frac{1}{p}} \times$$

$$\times \sum_{l=1}^{n} \Big[\int_{\tau=t_0}^{t} (R_l^\circ(\tau))^q \Big]^{1/q} \Big\} \exp\Big(-\frac{1}{2}\int_{t_0}^{t} \psi_k(\xi)\, d\xi\Big) = x_s^{**} \qquad (s=1,\ldots,n) \tag{3.21}$$

The conditions of stability are

$$x_s^{**} \leqslant x_s^\circ \qquad (s=1,\ldots,n) \qquad (t_0 \leqslant t \leqslant T) \tag{3.22}$$

In the case when the coefficients change slowly, i.e. when (2.1) holds, we can use the results of Section 2 for the estimates of x_s, and thus obtain the corresponding estimates for $|x_s^*|$.

4. Nonlinear system. Let us consider the sytem

$$\frac{dx_s}{dt} = p_{s1}(t)x_1 + \cdots + p_{sn}(t)x_n + X_s \qquad (s = 1, \ldots, n) \tag{4.1}$$

where $X_s = X_s(t, x_1, \ldots, x_n)$ are holomorphic functions of the variables $x_1, \ldots, x_n$ whose expansion in terms of these variables begin with terms of degree not lower than the second; the coefficients of powers of x_s are real, continuous, bounded function of t.

If the roots of the characteristic equation of the first approximation of system (4.1) coincide with the roots of the characteristic equation of Section 1, one can reduce the system (4.1) with the aid of the transformation (1.2) and (1.3) to the form

$$\begin{aligned} \frac{dy_j}{dt} &= \varkappa_j y_j + Q_j + Y_j \qquad (j = 1, \ldots, m) \\ \frac{dy_i}{dt} &= \lambda_i y_i - \mu_i y_{\sigma+i} + Q_i + Y_i \\ \frac{dy_{\sigma+i}}{dt} &= \lambda_i y_{\sigma+i} + \mu_i y_i + Q_{\sigma+i} + Y_{\sigma+i} \qquad (i = m+1, \ldots, n-\sigma) \end{aligned} \tag{4.2}$$

Here $Y_s = a_{s1}X_1 + \ldots + a_{sn}X_n$ $(s = 1, \ldots, n)$ and Q_s have the same meaning as in Section 1.

If we restrict the discussion to the region

$$t \geqslant t_0, \qquad |x_s| \leqslant h \qquad (s = 1, \ldots, n) \tag{4.3}$$

where h is sufficiently small, then we obtain the same estimates as (1.19), since, in this case the sign of dV/dt does not depend on the terms containing Y_s. If, furthermore, the initial disturbances (1.15) are sufficiently small then the conditions of stability (1.20) will be valid also for system (4.1).

5. Examples. Let us consider the system

$$\begin{aligned} \frac{dx_1}{dt} &= \left(-\frac{1}{2} + \varepsilon a \cos 2t\right)x_1 + (1 - \varepsilon a \sin 2t)\,x_2 \\ \frac{dx_2}{dt} &= (-1 - \varepsilon a \sin 2t)\,x_1 + \left(-\frac{1}{2} - \varepsilon a \cos 2t\right)x_2 \end{aligned} \tag{5.1}$$

Here, a is some positive number, ϵ is a parameter. We are given the following conditions

$$|x_{10}| \leqslant x_{10}^\circ, \qquad |x_{20}| \leqslant x_{20}^\circ \quad \text{when } t = t_0$$
$$|x_1| \leqslant x_1^\circ, \qquad |x_2| \leqslant x_2^\circ \quad \text{when } t \text{ lies in interval } [t_0, T]$$

where T is a given small number. Let us find the conditions of stability.

Suppose $t_0 = 0$. In this case we have

$$\frac{dx_1}{dt}=\left(-\frac{1}{2}+\varepsilon a\right)x_1+x_2+\varepsilon a\,(1-\cos 2t)\,x_1-(\varepsilon a\sin 2t)\,x_2.$$
$$\frac{dx_2}{dt}=-x_1+\left(-\frac{1}{2}-\varepsilon a\right)x_2-\varepsilon a\sin 2t\,x_1+\varepsilon a\,(1-\cos 2t)\,x_2 \qquad (5.2)$$

If T and ϵ are very small, the investigation can be carried out in a manner analogous to that of Section 2.

If $(\epsilon a)^2 < 1$, the equation

$$\begin{vmatrix} \left(-\frac{1}{2}+\varepsilon a\right)-\varkappa & 1 \\ -1 & \left(-\frac{1}{2}-\varepsilon a\right)-\varkappa \end{vmatrix}=0 \qquad (5.3)$$

has the roots

$$\varkappa_1=-\frac{1}{2}+i\sqrt{1-(\varepsilon a)^2},\qquad \varkappa_2=-\frac{1}{2}-i\sqrt{1-(\varepsilon a)^2}$$

In this case we have the transformations:

$$y_1=\varepsilon a x_1+x_2,\qquad y_2=\sqrt{1-(\varepsilon a)^2}\,x_1 \qquad (5.4)$$

$$x_1=\frac{y_2}{-\sqrt{1-(\varepsilon a)^2}},\qquad x_2=y_1+\frac{\varepsilon a}{-\sqrt{1-(\varepsilon a)^2}}\,y_2 \qquad (5.5)$$

By means of these transformations we reduce the system (5.2) to the form

$$\frac{dy_1}{dt}=-\frac{1}{2}\,y_1-\sqrt{1-(\varepsilon a)^2}\,y_2+\varepsilon Q_1$$
$$\frac{dy_2}{dt}=-\frac{1}{2}\,y_2+\sqrt{1-(\varepsilon a)^2}\,y_1+\varepsilon Q_2 \qquad (5.6)$$

Let us consider the function $V = e^{-a(t)}\,(y_1^2 + y_2^2)$. According to (2.22) and (2.24) we have $-a'(t) - 1 < 0$, and

$$e^{-\delta t}\,e^{t}\,(y_1^2+y_2^2)<y_{10}^2+y_{20}^2 \qquad (5.7)$$

Here δt is very small. Therefore, one may consider $e^{-\delta t} \approx 1$. By (5.4) we have

$$|\,y_{10}\,|\leqslant|\,\varepsilon a\,|\,x_{10}^{\circ}+x_{20}^{\circ},\qquad |\,y_{20}\,|\leqslant|\,\sqrt{1-(\varepsilon a)^2}\,|\,x_{10}^{\circ}$$

From this and from (5.7) it follows that

$$e^{t}\,(y_1^2+y_2^2)<(x_{10}^{\circ 2}+2\,|\,\varepsilon a\,|\,x_{10}^{\circ}x_{20}^{\circ}+x_{20}^{\circ 2})$$

The last inequality yields the estimates:

$$|y_1| < e^{-1/2\,t}(x_{10}^{\circ 2} + 2\,|\varepsilon a|\,x_{10}^{\circ}\,x_{20}^{\circ} + x_{20}^{\circ 2})^{1/2}$$
$$|y_2| < e^{-1/2\,t}(x_{10}^{\circ 2} + 2\,|\varepsilon a|\,x_{10}^{\circ}x_{20}^{\circ} + x_{20}^{\circ 2})^{1/2}$$

Because of (5.5) we have

$$|x_1| < \frac{1}{|\sqrt{1-(\varepsilon a)^2}|}\, e^{-1/2\,t}(x_{10}^{\circ 2} + 2\,|\varepsilon a|\,x_{10}^{\circ}x_{20}^{\circ} + x_{20}^{\circ 2})^{1/2} = x_1^*$$
$$|x_2| < \left(1 + \frac{\varepsilon a}{|\sqrt{1-(\varepsilon a)^2}|}\right) e^{-1/2\,t}\,(x_{10}^{\circ 2} + 2\,|\varepsilon a|\,x_{10}^{\circ}x_{20}^{\circ} + x_{20}^{\circ 2})^{1/2} = x_2^*$$

The conditions of stability will be

$$x_1^* \leqslant x_1^{\circ}, \qquad x_2^* \leqslant x_2^{\circ} \qquad (t_0 \leqslant t \leqslant T)$$

Thus, for example, if

$$x_{10}^{\circ} = 1, \qquad x_{20}^{\circ} = 1; \qquad x_1^{\circ} = 2, \qquad x_2^{\circ} = 2$$

then, if $|\epsilon a| \leqslant 1/2$, the conditions of stability will be fulfilled.

2. An example of B.V. Bulgakov's. Bulgakov evaluated the "accumulated disturbances" of a system with variable coefficients by his own method on the basis of three systems with constant coefficients:

$$\frac{dx_1}{dt} = 0.6x_1 + 2.7x_2 + R_1, \qquad \frac{dx_2}{dt} = -1.5x_1 - x_2 + R_2 \text{ on } [0,4] \tag{5.8}$$
$$\frac{dx_1}{dt} = x_1 + 3.3x_2 + R_1, \qquad \frac{dx_2}{dt} = -1.9x_1 - 1.3x_2 + R_2 \text{ on } [4.7] \tag{5.9}$$
$$\frac{dx_1}{dt} = 0.9x_1 + 3.8x_2 + R_1, \qquad \frac{dx_2}{dt} = -2.4x_1 - 1.1x_2 + R_2 \text{ on } [7.10] \tag{5.10}$$
$$(|R_1| \leqslant l_1 = \text{const}, \qquad |R_2| \leqslant l_2 = \text{const})$$

Here, when $t_0 = 0$, we have $x_{10} = 1$, $x_{20} = 0$.

It is required to estimate x_1 when $t = 10$. Bulgakov obtained the result

$$x_1 \leqslant 0.1362 + 3.8777 l_1 + 5.7620 l_2 \tag{5.11}$$

For the sake of simplicity we consider in place of the systems (5.8), (5.9) and (5.10) a single system in the interval [1,10] :

$$dx_1/dt = 0.8x_1 + 3.21x_2 + R_1, \qquad dx_2/dt = -1.89x_1 - 1.18x_2 + R_2 \tag{5.12}$$

The characteristic equation of the system (3.12) will be

$$\begin{vmatrix} 0.8 - \varkappa & 3.21 \\ -1.89 & -1.18 - \varkappa \end{vmatrix} = 0$$

It has the roots $\kappa = -0.19 \pm i\,2.25$.

The direct and inverse linear transformations are respectively

$$y_1 = \frac{0.99}{3.21} x_1 + x_2, \quad y_2 = \frac{2.25}{3.20} x_1, \quad x_1 = \frac{3.21}{2.25} y_2, \quad x_2 = y_1 - \frac{0.99}{2.25} y_2$$

The estimates for $|y_1|$ and $|y_2|$ will be

$$|y_1| \leqslant e^{-0.19t} (y_{10}^2 + y_{20}^2)^{0.5} = e^{-0.19t} \cdot 0.766 \qquad |y_2| \leqslant e^{-0.19t} 0.766$$

When $t = 10$, we have

$$|x_1| \leqslant \frac{3.21}{2.25} e^{-1.9} 0.766 = 0.162$$

Let us find the estimate of the particular integral u_1 of system (5.12). For the fundamental system of solutions we have the following initial conditions:

$$x_{10}^{(1)} = 1, \quad x_{20}^{(1)} = 0, \quad x_{10}^{(2)} = 0, \quad x_{20}^{(2)} = 1$$

From this and from equation (3.14) we obtain

$$(y_{10}^{(1)^2} + y_{20}^{(1)^2})^{1/2} = 0.766, \qquad (y_{10}^{(2)^2} + y_{20}^{(2)^2})^{1/2} = 1$$

Therefore

$$|u_1| \leqslant \frac{3.21}{2.25} e^{-1.9} \cdot 1 \cdot 10 (l_1 + l_2) = 2.12 l_1 + 2.12 l_2$$

Finally we obtain

$$|x_1^*| \leqslant 0.162 + 2.12 l_1 + 2.12 l_2 \tag{5.13}$$

This expression (5.13) can be compared with (5.11).

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